J. Fluid Mech. (1986), *vol.* 163, *pp.* 21–26
Printed in Great Britain

Remarks on a paper by Aref and Flinchem

By R. T. PIERREHUMBERT
Geophysical Fluid Dynamics Laboratory/N.O.A.A., Princeton University, Princeton, N.J.

(Received 9 April 1985 and in revised form 13 August 1985)

It is shown that the vortex filament in background shear considered in Aref & Flinchem (1984) is unstable to infinitesimal disturbances, and that the numerical results described therein are consistent with the characteristics of the instability.

Aref & Flinchem (1984, hereinafter referred to as AF) have presented numerical integrations of the nonlinear equations governing the motion of a vortex filament in an imposed background shear flow, within the local induction approximation. They find that initially localized disturbances disperse into planar undulations of the filament and imply that this is a result of intrinsically nonlinear aspects of soliton breakup. AF propose that this is a new explanation of the instability observed in the free-shear-layer experiments of Breidenthal (1979) and contrast their mechanism with earlier theories based on linear vortex-tube instability (notably that of Pierrehumbert & Widnall 1982, hereinafter referred to as PW). In the following we show that, on the contrary, the system studied by AF is unstable to a mode similar to the translative instability discussed in PW, and that many of the results of AF can be viewed as manifestations of this instability.

We shall adhere to the notation of AF, except as otherwise specified. Consider a straight filamentary vortex perturbed by a sinusoidal disturbance:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} + \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ 0 \end{bmatrix} \exp(\mathrm{i}(\alpha z - \omega t)). \tag{1}$$

Substituting into equation (7) of AF and linearizing, we obtain

$$-\mathrm{i}\omega\tilde{x} = \alpha^2 C\Gamma\tilde{y} + U'\tilde{y}, \quad -\mathrm{i}\omega\tilde{y} = -\alpha^2 C\Gamma\tilde{x}, \tag{2a, b}$$

where C is a positive constant and Γ is the circulation about the filament. The first terms in ($2a$, b) yield the well-known self-induced rotation, and are equivalent to the formula given in Bachelor (1967). The second term in (2) arises from the advection by the external shear field $U_{\mathrm{ext}}(y)$, which has shear $U' = \mathrm{d}U_{\mathrm{ext}}/\mathrm{d}y|_{y=0}$ at the location of the undisturbed vortex. (In the notation of AF, $U' = U_0/\Delta$.) In accordance with the right-hand rule, the circulation of the background shear has the same sign as $-U'$; relevance to the free shear layer therefore requires $-U'/\Gamma > 0$, so that the vortices have circulation of the same sign as that of the background shear. (AF state that their calculations were carried out with $C\Gamma = 1$, but Aref (personal communication) has informed us that $C\Gamma = -1$ was actually used, as is appropriate for $U' > 0$.) The eigenvalues and corresponding eigenvectors of (2) are given by

$$\omega = \pm C\Gamma[\alpha^2(\alpha^2 + U'/C\Gamma)]^{\frac{1}{2}}, \quad \tilde{y}/\tilde{x} = -\frac{\pm\mathrm{i}|\alpha|}{(\alpha^2 + U'/C\Gamma)^{\frac{1}{2}}}. \tag{3a, b}$$

The system is thus unstable when $-U'/C\Gamma > 0$, as in the free shear layer. The unstable band of wavenumbers is $0 < \alpha^2 < -U'/C\Gamma$, and the maximum instability occurs at

$$|\alpha| = \left(\frac{-U'}{2C\Gamma}\right)^{\frac{1}{2}}. \tag{4}$$

AF note that dimensional considerations require that the lengthscale of the undulations appearing on the filament be proportional to $(\Gamma/U')^{\frac{1}{2}}$ (as in (4)), and confirm that this scaling holds over a broad range of simulations. To evaluate the utility of the stability theory, it therefore suffices to check (4) against any one of the numerical simulations. The calculation exhibited in figure 7 of AF offers the cleanest possibility for comparison, as the integration has proceeded long enough for the wave packet to have attained a fairly monochromatic form, yet not so long as to have been contaminated by boundary effects. For the parameters appropriate to this case, (4) predicts a wavelength of 0.314. This is reasonably close to the wavelength 0.26 estimated by measuring the distance between zero crossings in figure 7. It does not seem likely that the difference between these values can be accounted for on the basis of truncation error. More probably, nonlinear effects are responsible for the modest shift in wavelength. Specifically, the curvature of a planar curve $f(z)$ is given by $(\mathrm{d}^2f/\mathrm{d}z^2)/(1+(\mathrm{d}f/\mathrm{d}z)^2)^{\frac{3}{2}}$, whereas in linear theory the denominator is approximated by unity. Nonlinearity thus reduces the overall curvature as compared to the estimate used in linear theory, and therefore tends to reduce the self-induced rotation rate. A shorter wavelength is then required to attain the rotation rate assumed in linear theory, in consequence of which the most rapidly growing disturbance occurs at a smaller scale.

From (3*b*), the most unstable mode lies in a plane tilted at a 45° angle to the (x, z)-plane, in accord with figure 11 of AF. The mode is in this regard quite similar to the translative instability discussed in PW. Another feature in common with the translative mode is that the growth rate approaches zero linearly as $\alpha \rightarrow 0$.

The physical mechanism of the instability is essentially the same as that discussed by Moore & Saffman (1971) for the case of a vortex filament in an irrotational plane strain. An important difference is that the sheared case has a long-wave cutoff, whereas the strained case does not. This occurs for the simple reason that the position of a point vortex displaced from its equilibrium position in a plane strain diverges exponentially with time, whereas that of a point vortex in a shear flow does not. In the latter case, when $-U'/\Gamma > 0$ the self-induced rotation is *opposite* to the background circulation and at finite wavenumbers advects the vortex into regions of ever greater background velocity, leading to exponential growth. The cooperative effect of self-induced rotation and shear, combined with the fact that a rapidly rotating disturbance averages out the effects of the shear, leads to the finite preferred wavenumber.

AF also claim that nonlinear dynamics is necessary for the creation of a sinusoidal wavetrain from an initially localized disturbance, and imply that the linear instability theories fail in this regard. This is not so. Consider the evolution of a field variable $f(z, t)$ in a system supporting normal modes with purely real growth rate $g(\alpha)$ having a maximum at $\alpha = \alpha_0$. By means of the method of steepest descent it can be shown that the initial condition $f(z, 0) = \delta(z)$ leads to a long-time asymptotic form of $f(z, t)$ which is proportional to

$$\frac{\exp\left[g(\alpha_0)\,t + \mathrm{i}\alpha_0 z\right] \exp\left(z^2/2g''(\alpha_0)\,t\right)}{(-g''(\alpha_0)\,t)^{\frac{1}{2}}}. \tag{5}$$

This has the form of a sinusoidal wave with wavenumber equal to that of the most unstable wave, modulated by a Gaussian envelope whose width grows like $t^{\frac{1}{2}}$. One finds essentially this pattern in figure 7 in AF. Similar results to (5) can be obtained when g is complex, and the asymptotic form is typically attained rather quickly (for a review of the dynamics of unstable wave packets, see Pierrehumbert 1984).

It is nonetheless instructive to consider the detailed predictions of linear theory for the evolution of a soliton-like disturbance to a vortex filament. This exercise will help to isolate the effects of nonlinearity. Through application of Fourier analysis an arbitrary initial disturbance can be synthesized from eigenmodes of the form given in (3*b*), whereafter the time evolution of each component is given by (3*a*). The shape of the filament at any subsequent time is then determined by transforming back to physical space. We have carried out this procedure for a localized helical initial disturbance described by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(\alpha_0 z) \\ \sin(\alpha_0 z) \end{bmatrix} \exp(-\lambda z^2). \tag{6}$$

The detailed shape of this disturbance is somewhat different from that of the solitons discussed in AF, but (6) is simpler to work with as its Fourier transform can be carried out analytically. The transformation back to physical space for $t > 0$ was effected by means of a numerical Fourier transform. Results for $\alpha_0 = -80$, $\lambda = 450$ (yielding initial scales similar to figure 7*a* of AF) are shown in figures 1 and 2. These calculations were carried out in an essentially unbounded z domain.

It is evident from figure 1 that, following a brief adjustment period, the disturbance achieves a form qualitatively like that described by (5); the subsequent evolution bears a clear resemblance to the corresponding results in AF. The three-dimensional structure of the disturbance is illustrated in figure 2, and is characterized by a rapid evolution of the initially helical perturbation into the planar form characteristic of the most unstable eigenmode. Despite the overall similarity, there are three points of disparity with the nonlinear results which merit comment. The first is that the ultimate wavelength of the disturbance is somewhat greater in linear than in nonlinear theory: this has been noted and discussed above. The second is that the peak of the unstable wave packet in linear theory is located at the site of the initial disturbance, whereas in the nonlinear problem it is shifted slightly to the right. This suggests a relic of soliton behaviour, in that the initial disturbance in the latter case appears to move to the right a bit before breaking up into unstable waves. The third, and most striking, difference is that the initial-adjustment stage in linear theory involves the radiation of short helical waves to the right (figures 1*c* and 2*b*). This is readily understood in terms of (3), which implies that shortwave disturbances are highly dispersive and have group velocities increasing in proportion to wavenumber. Short, helical waves with a clockwise twist propagate toward positive z, while those with a counterclockwise twist propagate toward negative z; the dominant rightward propagation in figure 1 is a consequence of the sign of α_0 in the initial condition. Shortwave radiation of this sort is conspicuously absent in the nonlinear results. Soliton dynamics almost certainly plays a role here, preventing dispersion by balancing it against nonlinearity in the classical fashion; note particularly that small-scale motions are relatively unaffected by the shear, and are thus most likely to inherit some features of the soliton behaviour of the unsheared system.

On the basis of the instability theory, we hazard two predictions concerning the behaviour of the system studied in AF. The first prediction is that the soliton form of the initial condition is not essential to the generation of regular planar undulations;

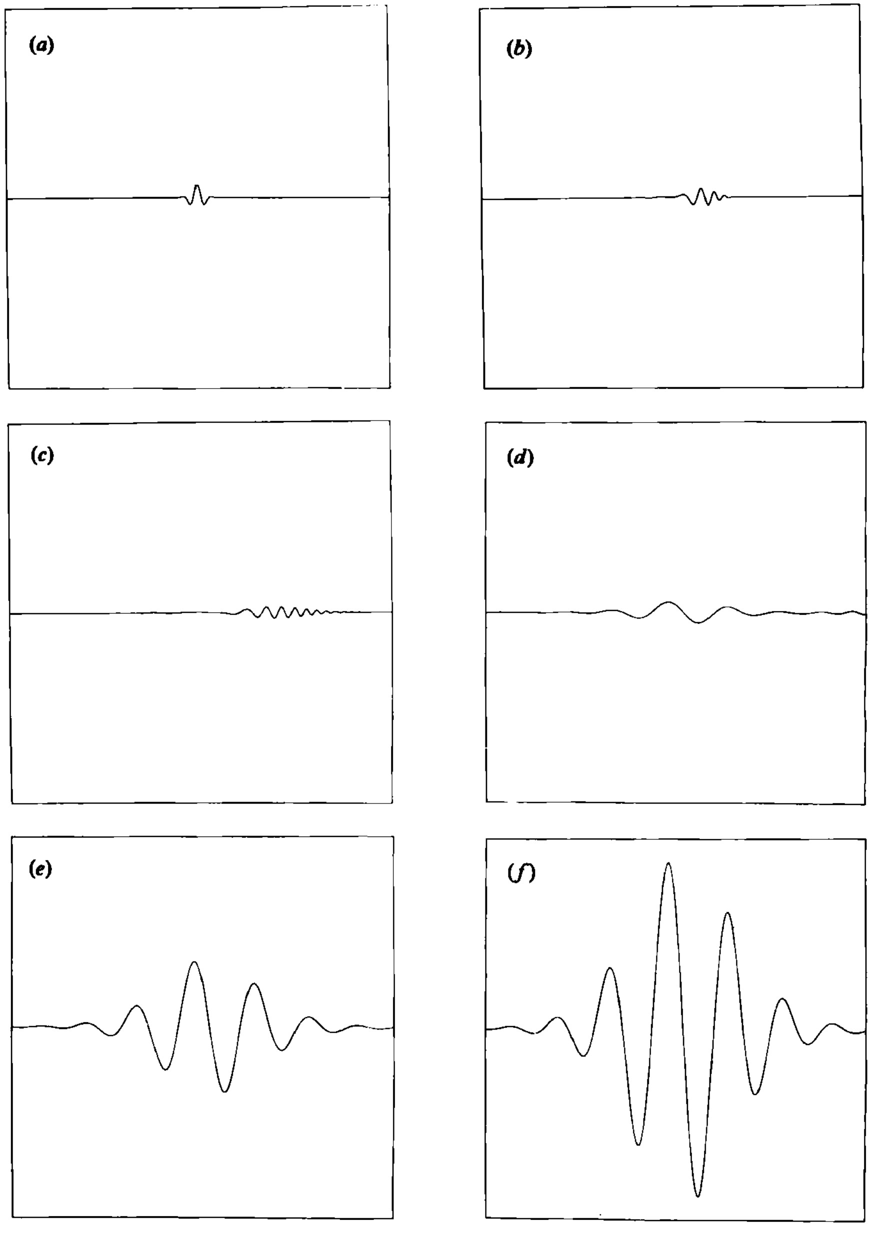

FIGURE 1. Time evolution of the perturbed filament, as determined by the linearized equations of motion. The filament position is shown projected on the (z, x)-plane with abscissa z ranging from -1 to 1, and ordinate x in arbitrary units. Parameter values are $U' = 800$, $C\Gamma = -1$, as in figure 7 of AF. Times are: (*a*) 0; (*b*) 0.001; (*c*) 0.0025; (*d*) 0.01; (*e*) 0.015; (*f*) 0.0175.

a localized planar perturbation with suitable tilt should serve as well. The second prediction is that the sharp wavelength selection and rapid growth of disturbance amplitude would be eliminated if the sign of the shear were reversed (yielding $U'/\Gamma > 0$). Some amplification of the initial disturbance is still expected in this case, even though the system is not exponentially unstable. According to (3*b*) $\tilde{x}/\tilde{y}$ is large for long waves, so that a small perturbation initially in the (y, z)-plane attains large amplitude as it rotates into the (x, z)-plane. The effect is limited to scales $\alpha^2 \ll U'/C\Gamma$.

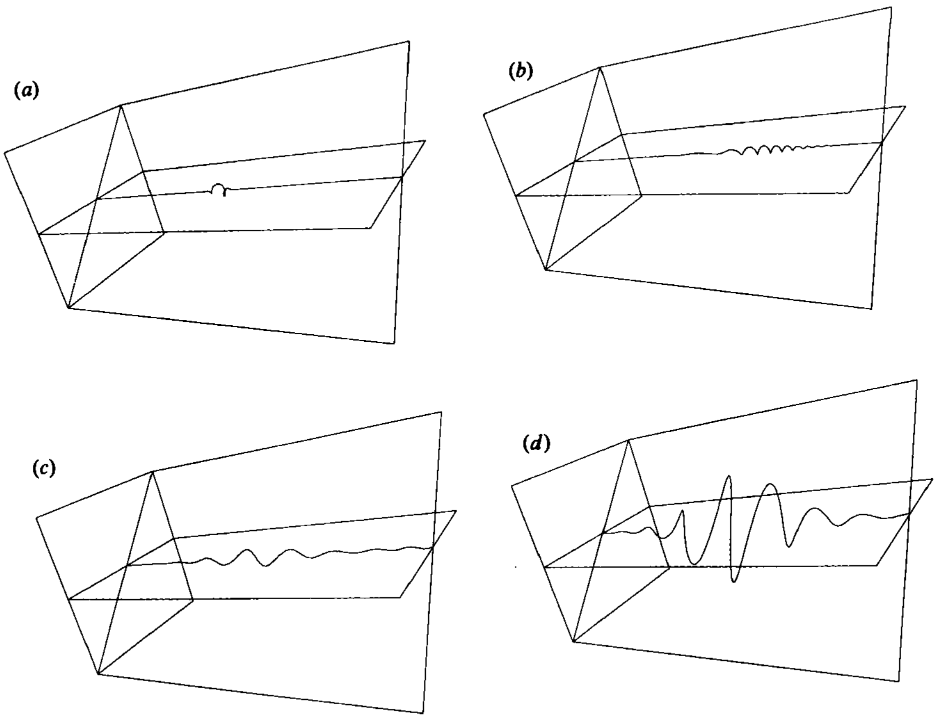

FIGURE 2. Perspective views corresponding to selected times in figure 1. The view is from above the shear layer, with the observer looking in the downstream direction; $z = 1$ is located at the right of the figure. The oblique plane extends from $(x, y) = (-1, -1)$ to $(x, y) = (1, 1)$. Times are: (*a*) 0; (*b*) 0.0025; (*c*) 0.01; (*d*) 0.015.

Although the calculations reported in AF have many fascinating features, it appears that they do not provide an explanation of the 'wiggle' instability observed in the experiments of Breidenthal (1979) that is distinct from the mechanisms which have been proposed earlier. Our claim is not that evolution described in AF is utterly unaffected by nonlinearity; rather, we assert that the development is best viewed in terms of the familiar scenario of a linear instability modified in its mature stage by nonlinearity. In fact, the results of AF to some extent support the conjectures made in PW concerning the nonlinear fate of the translative instability, insofar as the filament instability discussed above can be regarded as a phenomenological model of the translative instability described in PW. Such an identification is suggested by the fact that the Stuart vortices, whose stability was considered in PW, also consist of (relatively) concentrated vortex cores embedded in a background shear flow. This model suggests that the translative instability would disappear if all the vorticity were concentrated in cores of small radius, leaving the background field irrotational. Robinson & Saffman (1982), who investigated the stability of a linear array of corotating vortex filaments in irrotational flow, found no mode analogous to the translative instability; their finding is clearly consistent with the understanding of the translative instability based on the sheared-filament model.

An important caveat must be attached to the wavenumber selection exhibited in the filament instability described above. The sharp shortwave cutoff derives from the quadratic growth of the self-induced rotation with wavenumber characteristic of the local induction approximation. Even within the realm of validity of the filamentary approximation, this behaviour is not correct, as the 'constant' C in reality depends logarithmically on wavenumber (see e.g. equation (4.12) in Moore & Saffman 1971). Moreover, it is known that the filamentary approximation yields spurious results for wavelengths comparable to the core radius (see e.g. Tsai & Widnall 1976). Thus, it should not be surprising that the inviscid translative instability in PW does not exhibit the sharp shortwave cutoff seen in the filamentary model. Given the relatively shortwave nature of the observed instability, we suggest that the shortwave behaviour in the model of PW is more relevant than that of the filament model. This leaves open the question of the physical mechanism responsible for wavenumber selection in the experiments, though the filamentary model suggests that basic state flows with more concentrated cores and stronger background shears would yield sharper wavenumber selection.

REFERENCES

AREF, H. & FLINCHEM, E. P. 1984 Dynamics of a vortex filament in a shear flow. *J. Fluid Mech.* **148**, 477–497.

BATCHELOR, G. K. 1967 *Introduction to Fluid Dynamics*, p 511. Cambridge University Press.

BREIDENTHAL, R. 1979 Chemically reacting, turbulent shear layer. *AIAA J.* **17**, 310–311.

MOORE, D. W. & SAFFMAN, P. G. 1971 Structure of a line vortex in an imposed strain. In *Aircraft Wake Turbulence* (ed. J. H. Olsen, A. Goldburg & M. Rogers). Plenum Press.

PIERREHUMBERT, R. T. 1984 Local and global baroclinic instability of zonally varying flow. *J. Atmos. Sci.* **41**, 2141–2162.

PIERREHUMBERT, R. T. & WIDNALL, S. E. 1982 The two- and three-dimensional instabilities of a spatially periodic shear layer. *J. Fluid Mech.* **114**, 59–82.

ROBINSON, A. C. & SAFFMAN, P. G. 1982 Three-dimensional stability of vortex arrays. *J. Fluid Mech.* **125**, 411–427.

TSAI, C. Y. & WIDNALL, S. E. 1976 The stability of short waves on a straight vortex filament in a weak externally imposed strain field. *J. Fluid Mech.* **73**, 721.